Kac's question, planar isospectral pairs and involutions in projective space

2006 J. Phys. A: Math. Gen. 39 L385

(http://iopscience.iop.org/0305-4470/39/23/L02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) L385–L388 doi:10.1088/0305-4470/39/23/L02

LETTER TO THE EDITOR

Kac's question, planar isospectral pairs and involutions in projective space

Koen Thas[1]

Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, S22, B-9000 Ghent, Belgium

E-mail: kthas@cage.UGent.be

Received 15 March 2006, in final form 27 April 2006
Published 23 May 2006
Online at stacks.iop.org/JPhysA/39/L385

Abstract
In a paper published in *Am. Math. Mon.* (1966 **73** 1–23), Kac asked his famous question 'Can one hear the shape of a drum?'. Gordon *et al* answered this question negatively by constructing planar isospectral pairs in their paper published in *Invent. Math.* (1992 **110** 1–22). Only a finite number of pairs have been constructed till now. Further in *J. Phys. A: Math. Gen.* (2005 **38** L477–83), Giraud showed that most of the known examples can be generated from solutions of a certain equation which involves certain involutions of an n-dimensional projective space over some finite field. He then generated all possible solutions when $n = 2$. In this letter we handle all dimensions, and show that no other examples arise.

PACS number: 02.10.Ox
Mathematics Subject Classification: 37J10, 37N20, 43A85, 05B20, 05B25

1. Shape and sound of drums, according to Kac

In [6], M Kac formulated his famous question 'Can one hear the shape of a drum?'. Formally, this question amounts to finding *planar isospectral pairs*—non-isometric planar simply connected domains for which the sets $\{E_n \,\|\, n \in \mathbb{N}\}$ of solutions of the stationary Schrödinger equation

$$(\Delta + \mathbf{E})\Psi = 0$$

with $\Psi_{|\text{Boundary}} = 0$ are identical. Any example of such a pair of non-congruent planar isospectral domains yields a counter example to Kac's question.

Several counter examples were constructed to the analogous question on Riemannian manifolds—see, e.g., R Brooks [1]—but for *Euclidian domains*, that is, domains constructed

[1] The author is a Postdoctoral Fellow of the Fund for Scientific Research—Flanders (Belgium).

in the Euclidian affine plane $\mathbb{R}^2$ the question appeared to be a challenge for a long time period. Finally, C Gordon, D Webb and S Wolpert provided a pair of simply connected non-isometric Euclidian isospectral domains (which we will call 'planar isospectral pairs' or 'isospectral billiards' in this letter) in their seminal paper 'Isospectral plane domains and surfaces via Riemannian orbifolds' [4].

After *opus citatum*, other examples were found by various people—see especially the paper of P Buser, J Conway, P Doyle and K-D Semmler [2]—but essentially only a finite number of planar isospectral pairs are known at present.

Starting from a certain set of data in a projective space of dimension $n \geqslant 2$ over a finite field $\mathbf{GF}(q)$ with q elements, which we will call 'projective isospectral data', O Giraud [3] derives an equation that yields candidates for generating isospectral billiards. For $n = 2$, he solves the equation, and generates, by computer, all isospectral billiards that can arise.

In this letter, we classify all isospectral billiards that arise from projective isospectral data in any dimension, as described in the next section. Only the examples generated by Giraud arise eventually.

2. Projective isospectral data

In [3], O Giraud studies triples $(\mathbf{P}, \{\theta^{(i)}\}, r)$, where $\mathbf{P}$ is a finite projective space of dimension at least 2 [5], and $\{\theta^{(i)}\}$ a set of r non-trivial involutory automorphisms of $\mathbf{P}$, satisfying the following equation

$$r(|\mathbf{P}| - \mathrm{Fix}(\theta)) = 2(|\mathbf{P}| - 1), \tag{1}$$

for some natural number $r \geqslant 3$, where $\mathrm{Fix}(\theta) = \mathrm{Fix}(\theta^{(i)})$ is a constant number of fixed points of $\mathbf{P}$ under each $\theta^{(i)}$, and $|\mathbf{P}|$ is the number of points of $\mathbf{P}$. Call the triple $(\mathbf{P}, \{\theta^{(i)}\}, r)$ as above *projective isospectral data*.

The involutions $\theta^{(i)}$ have a certain permutation matrix representation $\mathbf{M}^{(i)}$, and given a triple as above, the incidence matrix $\mathbf{T}$ of $\mathbf{P}$ yields a 'transplantation matrix' (see [3]). If certain extra conditions on the $\mathbf{M}^{(i)}$ are satisfied, non-isometric planar isospectral pairs can be generated, each consisting of $|\mathbf{P}|$ copies of an r-sided polygonal base tile (see [3]). (For a pair of isospectral billiards on N copies of a tile with r sides, one needs r involutions acting on a set of N points with the property that the graph formed by joining each pair of points that are interchanged by at least one of the involutions has no loops and is connected. For any involution the number of edges is $(N - s)/2$, with s being the number of fixed points of the involution. The total number of edges must equal $N - 1$, and the group of transformations generated by the involutions must act transitively on the set of N points.)

For projective spaces of dimension 2—in other words *projective planes*—Giraud determined all solutions of equation (1), and showed that all examples of [2] can be generated by computer from the data obtained.

In a finite axiomatic projective plane, involutions can occur with a different number of fixed points. Indeed, if we consider, for example, the Desarguesian plane $\mathbf{PG}(2, q^2)$ (see the next section for a definition), then there are *Baer involutions* (also described in the next section), which have precisely $q^2 + q + 1$ fixed points, and *linear involutions*, which have $q^2 + 1 + \delta(q)$ fixed points, where $\delta(q) = 1$ if q is odd, and $\delta(q) = 0$ if q is even. However, O Giraud only considers involutions with the *same number* of fixed points, explaining the nature of the generalization considered in this letter. Involutions with a different number of fixed points will be handled in a subsequent paper (see also the final remark of the present letter).

In the next section, we solve equation (1) for any finite dimensional projective space over a finite field, and show that the solutions do not give rise to new examples.

3. Classification of planar isospectral pairs with projective isospectral data

Denote by $\mathbf{PG}(n,q)$, $n \in \mathbb{N} \cup \{-1\}$, the n-dimensional projective space over the Galois field $\mathbf{GF}(q)$ with q elements (q a prime power); we have $|\mathbf{PG}(n,q)| = \frac{q^{n+1}-1}{q-1}$. Note that $\mathbf{PG}(-1,q)$ is just the empty space.

We handle several cases of fixed points structures of involutions in the automorphism group $\mathbf{P\Gamma L}(n+1,q)$ of $\mathbf{PG}(n,q)$, according to Segre's classification [7].

3.1. Baer involutions

First suppose that θ is a *Baer involution*, that is, θ is not contained in the linear automorphism group of the space, so that q is a square, and θ fixes an n-dimensional subspace over $\mathbf{GF}(\sqrt{q})$ pointwise. It is convenient to denote $\mathbf{P}$ by $\mathbf{PG}(n,q^2)$ in this case, and the elementwise fixed space of θ by $\mathbf{PG}(n,q)$ for obvious reasons. Then (1) becomes

$$r\left(\frac{q^{2(n+1)}-1}{q^2-1} - \frac{q^{n+1}-1}{q-1}\right) = 2\left(\frac{q^{2(n+1)}-1}{q^2-1} - 1\right).$$

Whence

$$r(q^{2(n+1)} - q^{n+2} - q^{n+1} + q) = 2(q^{2(n+1)} - q^2).$$

It is clear that if $n \geqslant 2$ and $r \geqslant 3$, the latter expression does not have natural solutions. For, if one subtracts the right-hand side from the left-hand side, one obtains

$$q(q^n-1)[(r-2)(q^{n+1}-1) - 2(q+1)],$$

and since $r \geqslant 3$ and $q \geqslant 2$, we then have

$$(r-2)(q^{n+1}-1) - 2(q+1) \geqslant (r-2)(q^3-1) - 2(q+1) \geqslant q^3 - 2q - 3 \geqslant 1.$$

3.2. Linear involutions in even characteristic

If q is even, and θ is not of Baer-type, θ must fix an m-dimensional subspace of $\mathbf{PG}(n,q)$ pointwise, with $1 \leqslant m \leqslant n \leqslant 2m+1$. We obtain

$$r\left(\frac{q^{n+1}-1}{q-1} - \frac{q^{m+1}-1}{q-1}\right) = 2\left(\frac{q^{n+1}-1}{q-1} - 1\right),$$

implying

$$r(q^{n+1} - q^{m+1}) = 2(q^{n+1} - q).$$

Clearly, q^{m+1} divides $2q$, so $m = 1 = q-1$, and the only solution is given by $n = 2$ and $r = 3$. Examples of this type can be found in [2, 3].

3.3. Linear involutions in odd characteristic

If θ is a linear involution of $\mathbf{PG}(n,q)$, q odd, the set of fixed points is the union of two disjoint complementary subspaces. Denote these by $\mathbf{PG}(k,q)$ and $\mathbf{PG}(n-k-1,q)$, $k \geqslant n-k-1 > -1$. We suppose $n > 2$ since $n = 2$ is handled in [3]. We have to solve

$$r\left(\frac{q^{n+1}-1}{q-1} - \frac{q^{k+1}-1}{q-1} - \frac{q^{n-k}-1}{q-1}\right) = 2\left(\frac{q^{n+1}-1}{q-1} - 1\right),$$

or

$$(r-2)q^{n+1} + r + 2q = r(q^{k+1} + q^{n-k}). \tag{2}$$

Noting that $\frac{q^{n+1}}{2} > q^{n-k} + q^{k+1}$, an inductional argument on $r \geqslant 4$ leads to insolvability. Put $r = 3$, and note that q divides r. Then q must equal 3, and (2) is simplified to

$$3^{n+1} + 3 + 6 = 3(3^{k+1} + 3^{n-k}),$$

and clearly $n - k = 1$.

Consider three involutions $\theta^{(i)}$ in $\mathbf{PGL}(n+1, 3)$ (of $\mathbf{PG}(n, 3)$), $n \geqslant 3$, with axis a hyperplane and centre a point. When the data $(\mathbf{PG}(n, 3), \{\theta^{(i)}\}, 3)$ would generate planar isospectral pairs, we have to know that the latter are simply connected. One observes that this implies that $\langle\theta^{(1)}, \theta^{(2)}, \theta^{(3)}\rangle$ is a subgroup of $\mathbf{PGL}(n+1, 3)$ acting transitively on the points of $\mathbf{PG}(n, 3)$. But as $n > 2$, the axes of $\theta^{(1)}, \theta^{(2)}, \theta^{(3)}$ intersect non-trivially, so that each point of this intersection is fixed by $\langle\theta^{(1)}, \theta^{(2)}, \theta^{(3)}\rangle$, contradiction.

The main theorem is proved. □

4. Final remark

At present, the author is developing a general theory on planar isospectral pairs arising from *incidence isospectral data*, that is, from sets of involutory automorphisms with not necessarily the same fixed elements structure, of finite projective spaces and more general incidence geometries [8].

Acknowledgment

The author thanks an anonymous referee for some helpful remarks.

References

[1] Brooks R 1988 Constructing isospectral manifolds *Am. Math. Mon.* **95** 823–39

[2] Buser P, Conway J, Doyle P and Semmler K-D 1994 Some planar isospectral domains *Int. Math. Res. Not.* **9** approximately 9 pp

[3] Giraud O 2005 Finite geometries and diffractive orbits in isospectral billiards *J. Phys. A: Math. Gen.* **38** L477–83

[4] Gordon C, Webb D and Wolpert S 1992 Isospectral plane domains and surfaces via Riemannian orbifolds *Invent. Math.* **110** 1–22

[5] Hirschfeld J W P 1998 *Projective Geometries Over Finite Fields (Oxford Mathematical Monographs)* 2nd edn (New York: Clarendon/Oxford University Press)

[6] Kac M 1966 Can one hear the shape of a drum? *Am. Math. Mon.* **73** 1–23

[7] Segre B 1961 *Lectures on Modern Geometry. With an appendix by Lucio Lombardo-Radice (Consiglio Nazionale delle Ricerche Monografie Matematiche* vol 7*)* (Rome: Edizioni Cremonese)

[8] Thas K Incidence isospectral data, in preparation